Uniform Boundedness Principle

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Outline of the talk

Many of the most important theorems in analysis assert that pointwise hypotheses imply uniform conclusions. Perhaps the simplest example is the result that "**a continuous function on a compact set is uniformly continuous**".

The uniform boundedness theorem is one of the central theorems of functional analysis and it has first been published in Banach's thesis, in the year 1922. Uniform boundedness principle was discovered by Lebesgue in 1908 in investigations on Fourier series, it was isolated as a general principle by Banach and Steinhaus.

Outline of the talk

The uniform boundedness theorem (UBT) enables us to determine whether the norms of a given collection of bounded linear operators $\{T_\alpha\}$ have a finite least upper bound. Clearly, since the norm was defined to be a real-valued (not an extended real-valued) mapping, the norm of each T_α must be finite, but there is no guarantee that they might not form an increasing sequence. The uniform boundedness theorem provides a criterion for determining when such an increasing sequence is not formed. That is, it states that a pointwise bounded sequence of bounded linear operators on Banach spaces is also uniformly bounded.

In the talk, we discuss the following :

- Proof of the UBT using Baire's theorem
- Proof of the UBT using gliding hump argument
- Simple elementary proof of the UBT
- Consequences/applications of UBT

Notations

Before we start, let us see the notations first.

$\mathbb{K}$	the field of real or complex scalars
ℓ_1	the set of absolutely summable sequences
ℓ_2	the set of square summable sequences
ℓ_∞	the set of bounded sequences
c_0	the set of convergent sequences coverging to 0
B_X	the closed unit ball in X
S_X	the unit sphere in X
$\overline{M}$	the closure of M
X^*	the dual of X
$N(T)$	null space of T
$R(T)$	range of T
T^*	the adjoint of T
$\|T\|$	a norm of the operator T
$\mathcal{B}(X, Y)$	the space of bounded linear operators from X into Y

Pointwise and Uniformly Bounded

Definition 1.

*Let X and Y be normed spaces and let $\mathcal{A}$ be a family of linear operators from X into Y. The family $\mathcal{A}$ is said to be **pointwise bounded** on a subset E of X if for each $x \in E$, there exists $M_x > 0$ such that $\|Ax\| \leq M_x$ for all $A \in \mathcal{A}$, that is, $\sup_{A \in \mathcal{A}}\{\|Ax\|\} < \infty$. In other words, for each $x \in E$, there exists a ball $B[0, M_x]$ contains all Ax with $A \in \mathcal{A}$.*

*The family $\mathcal{A}$ is said to be **uniformly bounded** on a subset E of X if there exists $M > 0$ such that $\|Ax\| \leq M$ for all $A \in \mathcal{A}$ and for all $x \in E$. In other words, there exists a ball $B[0, M]$ contains all Ax with $A \in \mathcal{A}$ and $x \in E$.*

We simply call $\mathcal{A}$ is **uniformly bounded** (or, **norm-bounded**) when $\{\|A\| : A \in \mathcal{A}\}$ is a bounded subset of real numbers (that is, $\sup_{A \in \mathcal{A}}\{\|A\|\} < \infty$).

Clearly, uniformly bounded implies that pointwise bounded. The converse is also true in finite dimensional normed spaces.

Exercise 2.

If a collection $\mathcal{A}$ of linear operators on a finite dimensional normed linear space X which is pointwise bounded, then $\mathcal{A}$ is uniformly bounded.

Proposition 3.

Let X and Y be normed spaces and let $\mathcal{A}$ be a family of linear operators from X into Y. Let $E_0 = \{x \in X : \|x\| \leq, <, = r\}$.

- *$\mathcal{A}$ is pointwise (uniformly) bounded on E_0 iff $\mathcal{A}$ is pointwise (uniformly) bounded on X.*
- *$\mathcal{A}$ is uniformly bounded on X iff $\mathcal{A} \subseteq \mathcal{B}(X, Y)$ and $\{\|A\| : A \in \mathcal{A}\}$ is a bounded subset of real numbers, that is, $\sup_{A \in \mathcal{A}} \{\|A\|\} < \infty$.*

Pointwise and Uniformly Bounded

Theorem 4.

Let X be a complete metric space and $\mathcal{A}$ be a set of continuous complex-valued functions on X (not necessarily linear). Then either

1. *$\mathcal{A}$ is uniformly bounded on some open ball (i.e., there exists a nonempty open ball U in X such that $\sup\{|Ax| : A \in \mathcal{A}, x \in U\} < \infty$,*

   or
2. *there exists a dense subset D of X such that $\mathcal{A}$ is not ponitwise bounded at every point of D (i.e., $\sup\{|Ax| : A \in \mathcal{A}\} = \infty$ for each $x \in D$).*

Corollary 5.

Let (f_n) be a sequence of complex-valued continuous functions on the real line which converges at every point. Then there is an interval I and a finite real number M such that $|f_n(x)| < M$ for all $x \in I$ and $n = 1, 2, \ldots$.

Theorem 6 (Uniform boundedness principle, 1922).

Suppose X is Banach, Y is a normed space and $\mathcal{A} \subseteq \mathcal{B}(X, Y)$. If $\mathcal{A}$ is pointwise bounded, then $\mathcal{A}$ is uniformly bounded. That is, when X is Banach, uniform boundedness and pointwise boundedness are same.

Proof. For each $x \in X$, there exists $M_x > 0$ such that $\|Ax\| \leq M_x$ for all $A \in \mathcal{A}$. Let $E_n = \{x \in X : \|Ax\| \leq n, \forall A \in \mathcal{A}\}$. Then $X = \cup_{n=1}^{\infty} E_n$. By Baire's category theorem, at least one E_k is not nowhere dense set. That is, $(\overline{E_k})^\circ = E_k^\circ \neq \emptyset$. As E_k has nonempty interior, there exists $u \in E_k$ and $r > 0$ such that $B(u, r) \subseteq E_k$. For every nonzero $x \in X$ we have $u + \frac{x}{2\|x\|} \in B(u, r) \subseteq E_k \Rightarrow \|Au + A(\frac{x}{2\|x\|})\| \leq k$, for all $A \in \mathcal{A}$. Now $\|A(\frac{rx}{2\|x\|})\| \leq 2k$, for all $A \in \mathcal{A}$ since $\|Au\| \leq k$. Therefore $\|Ax\| \leq \frac{4k}{r}\|x\|$, for all $x \in X$, for all $A \in \mathcal{A}$. Thus $\|A\| \leq \frac{4k}{r}$ for all $A \in \mathcal{A}$.

Aliter.

Let $D_n = \{x \in X : \|Ax\| > n, \text{ for some } A \in \mathcal{A}\}$. Let $D = \cap_{n=1}^{\infty} D_n$. Since X is complete, Baire's theorem implies that either D is dense in X, or some D_n is nondense in X, that is, $\overline{D_N} \neq X$. Let D be dense in X, and $s(x) = \sup\{\|Ax\| : A \in \mathcal{A}\}$. If $x \in D_n$, then $s(x) \geq \|Ax\| > n$ for some $A \in \mathcal{A}$. Thus, if $x \in D$, then $x \in D_n$ for all n, and so $s(x) > n$ for all n, $s(x) = \infty$, a contradiction to pointwise bounded. Suppose some D_n is nondense in X, that is, $\overline{D_N} \neq X$. Then there is $a \in X$, with $a \notin \overline{D_N}$. Hence there is $r > 0$ such that the open ball $B(a, r)$ is disjoint from D_N. So if $x \in B(a, r)$, then $x \notin D_N$, and hence $x \notin D_N$ for all $A \in \mathcal{A}$, so $\|Ax\| \leq N$ for all $A \in \mathcal{A}$. Thus, $\|Ax\| \leq N$ for all $A \in \mathcal{A}$ and $x \in B = B(a, r)$.

By Proposition 3 and Theorem 6, we have the following:

Corollary 1.

Let X be a Banach space, Y be a normed space and $\mathcal{A}$ be a subset of $\mathcal{B}(X, Y)$ such that $\mathcal{A}$ is pointwise bounded. Then for each bounded subset E of X, the set $\mathcal{A}$ is uniformly bounded on E.

Completeness hypothesis cannot be dropped.

The completeness hypothesis cannot be dropped from uniform boundedness principle.

Example 7.

$X = c_{00}$ with respect to the norm $\|.\|_p$, $1 \le p \le \infty$. Define $f_n(x) = nx_n$. Then $\sup_x |f_n(x)| < \infty$ (because the terms of the sequence x are zero after some stage.) However $\|f_n\| = n$, and hence $\sup \|f_n\| = \infty$.

Example 8.

$X = c_{00}$ with sup norm and $f_n(x) = \sum_{j=1}^{n} x_j$. $\{f_n\}$ is pointwise bounded and $\|f_n\| = n$ because $|f_n(x)| \le n\|x\|_\infty$ and $f_n(1, \ldots, 1, 0, \ldots) = n$. Note that $\{f_n\}$ is not uniformly bounded because $\|f_n\| = n$ is an unbounded set of real numbers.

Example 9.

The linear space $X = \mathcal{P}[a, b]$, all polynomials on $[a, b]$ with respect to the norm $\|x\| = \max_{n=0}^{m}\{|a_n|\}$, where $x = a_0 + a_1 t + \cdots + a_m t_m$. Define $f_n(x) = \sum_{m=0}^{n} a_m$, then $\{f_n\}$ is pointwise bounded but it is not uniformly bounded, since $\{\|f_n\| = n + 1\}$ is unbounded.

Continuous Seminorm

Actually only the following properties of $A \in \mathcal{A}$ are used in the proof of Theorem 6. The function $x \mapsto \|Ax\|$ is continuous from X to nonnegative real numbers, $\|A(x+y)\| \leq \|Ax\| + \|Ay\|$ and $\|A(kx)\| \leq |k| \; \|Ax\|$ for all x and y in X and $k \in \mathbb{K}$.

A function having these properties is known as a **continuous seminorm** on X. Thus, the uniform boundedness theorem is true for a collection $\mathcal{A}$ of continuous seminorms on X.

Theorem 10.

Let X be a Banach space, and let $\{p_\lambda\}$ be a family of continuous nonnegative functions on X, each satisfying the conditions $p_\lambda(x+y) \leq p_\lambda(x) + p_\lambda(y)$ and $p_\lambda(-x) = p_\lambda(x)$. Suppose for each x, $\sup_\lambda p_\lambda(x) < \infty$. Then $\sup_\lambda \sup_{\|x\| \leq 1} p_\lambda(x) < \infty$.

Proof

For each n, let $C_n = \{x : \sup_\lambda p_\lambda(x) \leq n\} = \cap_\lambda \{x : p_\lambda(x) \leq n\}$. Since p_λ are continuous, C_n is closed. By the hypothesis $X = \cup_n C_n$. So, by the Baire category theorem there exists an n_0 such that the set C_{n_0} contains a closed ball $B[x_0, r]$. Let x be any element of X such that $\frac{1}{2}\|x\| \leq r$. Then the vectors $x_0 \pm \frac{x}{2}$ are in the ball $B[x_0, r]$. Since $x = x_0 + \frac{x}{2} - (x_0 - \frac{x}{2})$ we have
$p_\lambda(x) \leq p_\lambda(x_0 + \frac{x}{2}) + p_\lambda(x_0 - \frac{x}{2}) \leq 2n_0$.

This is true for all x with $\|x\| \leq 2r$. Hence, $\sup_\lambda \sup_{\|x\| \leq 2r} p_\lambda(x) \leq 2n_0 < \infty$. If $1 \leq 2r$, the proof is over. If this is not the case, choose a positive integer $m > \frac{1}{2r}$. Now if $\|x\| \leq 1$, then $\|\frac{x}{m}\| < 2r$, and $p_\lambda(x) \leq m p_\lambda(\frac{x}{m}) \leq 2mn_0$. So, $\sup_\lambda \sup_{\|x\| \leq 1} p_\lambda(x) \leq 2mn_0 < \infty$.

A version of the uniform boundedness principle for continuous 'affine' functions

A version of the uniform boundedness principle for continuous 'affine' functions defined on a bounded complete convex subset of a normed space X is as follows:

Theorem 11.

Let X (not necessarily Banach) and Y be normed spaces, E be a bounded complete convex subset of X, and $\mathcal{A}$ be a set of continuous maps (not necessarily linear) $A : E \to Y$ satisfying

$$A(rx + (1-r)y) = rAx + (1-r)Ay \quad \text{for } 0 < r < 1 \text{ and } x, y \in E,$$

such function is called an ***affine function****.*

Then a set $\mathcal{A}$ of continuous affine functions is uniformly bounded on E iff it is pointwise bounded on E.

We have the following theorem when a collection of bounded operators on Banach spaces is not uniformly bounded.

Theorem 12.

Let $\mathcal{A}$ be a family of bounded linear operators from a Banach space X to a normed space Y and D be the set of all $x \in X$ such that the set $S_x = \{\|Ax\| : A \in \mathcal{A}\}$ is unbounded. Then D is dense in X or empty.

Proof.

Since $\mathcal{A}$ is not uniformly bounded, it is not pointwise bounded. Then there exists $x_0 \in X$ such that $x_0 \in D$. Let $X_0 = X \backslash D$. As X_0 is a proper subspace of X ($x_0 \notin X_0$), the interior of X_0 is empty, (X_0 is nowhere dense, hence it is of first category). Therefore (the complement of X_0) D is dense in X.

Aliter.

Suppose that D is not dense in X. Then there is $a \in X$ with $a \notin \overline{D}$. So there is $r > 0$ such that the open ball $B(a, r)$ is disjoint from D. In particular, $a \notin D$. Let x be any element of X. We can find δ, with $0 < \delta < 1$ such that $\delta\|x - a\| < r$ (using this we are going to find another element not in D). Let $y = a + \delta(x - a)$. Then $\|y - a\| = \delta\|x - a\| < r$, and so $y \notin D$. Thus we see that the sets S_a and S_y with $x = a, y$ are bounded. So there exist finite constants M_1 and M_2 such that $\|Aa\| \leq M_1$, $\|Ay\| \leq M_2$, for all $A \in \mathcal{A}$. Since $Ay = Aa + \delta Ax - \delta Aa$, we get $\delta\|Ax\| = \|Ay\| + (\delta - 1)\|Aa\| \leq M_2 + (1 - \delta)M_1$ for all $A \in \mathcal{A}$. This implies that $x \notin D$. Since this is true for any $x \in X$, we see that D is empty.

Another Form.

Suppose a family of continuous linear operators on a Banach space X is not uniformly bounded with respect norm. **Then the set at which this family converges pointwise is of the first category** (i.e., it is a countable union of nowhere dense sets). Note that every closed subspace is of the first category.

Geometrically, the uniform boundedness theorem says that either each $A \in \mathcal{A}$ maps a bounded subset of a Banach space X into a fixed ball in Y or there is some $x \in X$ such that no ball in Y contains all Ax with $A \in \mathcal{A}$. The choice of such x is dense in X.

Theorem 13 (Banach-Steinhauss theorem).

Let X be a Banach space and Y be a normed space. Let I be an arbitrary indexing set and, for each $i \in I$, set $T_i \in \mathcal{B}(X, Y)$. Then, either there exists $M > 0$ such that $\|T_i\| \leq M$ for all $i \in I$, or $\sup_{i \in I} \|T_i x\| = \infty$ for all x belonging to some dense G_δ set in X.

Proof

For each $x \in V$, set $\psi(x) = \sup_{i \in I} \|T_i x\|$. Let $V_n = \{x \in V : \psi(x) > n\}$. Since each T_i is continuous and since the norm is a continuous function, it is easy to see that V_n is open for each n. Assume now that there exists N such that V_N fails to be dense in V. Then there exists $x_0 \in V$ and $r > 0$ such that $x + x_0 \notin V_N$ if $\|x\| < r$. (In other words, there is an open ball $B(x_0, r)$ centered at x_0 and of radius r, which does not intersect V_N.) This implies that $\psi(x + x_0) \leq N$ for all such x and so, for all $i \in I$, $\|T_i(x + x_0)\| \leq N$. Thus, if $\|x\| \leq \frac{r}{2}$, we have, for all $i \in I$, $\|T_i x\| \leq \|T_i(x + x_0)\| + \|T_i(x_0)\| \leq 2N$. It follows from this that, for all $i \in I$, $\|T_i\| \leq \frac{4N}{r}$ and so the first alternative holds with $M = \frac{4N}{r}$. The other possibility is that each V_n is dense, and so, V being complete, by Baire's theorem, $\cap_n V_n$ is a dense G_δ and for each $x \in \cap_n V_n$, we have $\psi(x) = \infty$.

Application of Uniform Boundedness Principle - Resonance Theorem

Theorem 14.

Let X be a Banach space, Y a normed space and $T_n \in \mathcal{B}(X, Y), n = 1, 2, \ldots$ such that $\sup_n \|T_n\| = \infty$. Then there exists an $x_0 \in X$ such that $\sup_n \|T_n x_0\| = \infty$.

*The point x_0 is often called a **point of resonance**. The points of resonance form a dense subset of X.*

Theorem 15.

Let X be a normed space and E be a subset of X. Then E is bounded iff $x^*(E)$ is bounded for every $x^* \in X^*$.

Proof

As E is bounded, there exists $k > 0$ such that $\|x\| \leq k$ for all $x \in E$. For each $x^* \in X^*$, $|x^*(x)| \leq \|x^*\| \, \|x\| \leq c\|x^*\|$, $x^*(E)$ is bounded. Or, continuous image of a bounded set is bounded, $x^*(E)$ is bounded, for each $x^* \in X^*$.

Conversely, X can be embedded into X^{**} by the canonical embedding $J : X \to X^{**}$ by $x \mapsto \phi_x$, where $\phi_x(x^*) = x^*(x)$. And $\|x\| = \|\phi_x\|$ (to prove this we used Hahn-Banach theorem). Through this isometric isomorphism J, we can consider X as a subspace of X^{**}. Since $\{|x^*(x)| : x \in E\}$ is bounded for each $x^* \in X^*$, $\{|\phi_x(x^*)| : x \in E\}$ is bounded for each $x^* \in X^*$.

Since X^* is Banach and $\{\phi_x(x^*)\}$ is pointwise bounded on X^*, by uniform boundedness theorem, $\{\phi_x : x \in E\}$ is uniformly bounded. Hence $\{\|\phi_x\| : x \in E\} = \{\|x\| : x \in E\}$ is bounded. The boundedness of $x^*(E)$ for every $x^* \in X^*$ implies the boundedness of E in X.

To check the boundedness of a set V in a Banach space, it thus suffices to verify that its image under each continuous linear functional is bounded. In finite dimensional spaces, this is what we precisely do. We check that the image under each coordinate projection is bounded and these form a basis for the dual space.

In the language of weak topologies, the conclusion of the Theorem 15 is read as "weakly bounded $\iff$ bounded".

Corollary 16.

Every weakly convergent sequence (x_n) is bounded and $\|x\| \leq \liminf \|x_n\|$.

Proof.

Suppose (x_n) is a weakly convergent sequence. Then for each $f \in X^*$, the sequence $(f(x_n))$ is convergent, bounded and hence (x_n) bounded.

Boundedness is a necessary condition for weakly convergent sequence. An unbounded sequence cannot be weakly convergent (or convergent).

Corollary 17.

Let X be a Banach space, Y be a normed space. Then $\mathcal{A} \subseteq \mathcal{B}(X, Y)$ is bounded iff $\{y^*(Tx) : T \in \mathcal{A}\}$ is bounded for every $x \in X$ and every $y^* \in Y^*$.

Proof.

For a fixed $x \in X$, $\{Tx : T \in \mathcal{A}\}$ is bounded, by resonance theorem. From Banach-Steinhaus theorem $\{T : T \in \mathcal{A}\}$ is bounded.

Theorem 15 is not true in general when the boundedness is replaced by convergence.

The convergence of $x^*(x_n)$ for every $x^* \in X^*$ may not imply the convergence of (x_n) in X.

Example 18.

$X = \ell_2 = X^*$ and $y = (x^*(e_1), x^*(e_2), \ldots)$. Then $y = (x^*(e_n)) \in (\ell_2)^* = \ell_2$. Hence $(x^*(e_n))$ converges to 0 for every $x^* \in \ell_2$. But (e_n) does not converge in X, since $\|e_n - e_m\|_2 = \sqrt{2}$, for all $n \neq m$.

Theorem 19.

Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. Then T is continuous iff $y^* \circ T \in X^*$ for every $y^* \in Y^*$ (the composition map $g \circ A : X \to \mathbb{K}$ belongs to X^* for every $g \in Y^*$).

Proof.

Since T is given to be linear, we see that $y^* \circ T$ is linear for each $y^* \in Y^*$. If T is continuous, $y^* \circ T$ is continuous. Conversely, suppose $y^* \circ T$ is continuous for each $y^* \in Y^*$. Let $S = \{x \in X : \|x\| \leq 1\}$. Then $(y^* \circ T)(S)$ is bounded in $\mathbb{K}$. Since this is true for all $y^* \in Y^*$, by uniform boundedness principle, $T(S)$ is bounded in Y, that is, T is bounded.

If the scalar field $\mathbb{K}$ is $\mathbb{R}$, we can give some geometric meaning to the resonance theorem.

Let X be a normed space over $\mathbb{R}$ and f be a linear functional on X. Corresponding to $\alpha \in \mathbb{R}$, consider the hyperplane

$$H_{f,\alpha} = \{x \in X : f(x) = \alpha\} = x_\alpha + N(f),$$

where $x_\alpha \in X$ is such that $f(x_\alpha) = \alpha$.

Now, for $E \subseteq X$, the set $f(E)$ is bounded if and only if there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq f(x) \leq \beta$ for all $x \in E$ iff E is on the left side of $H_{f,\beta}$ and on the right side of $H_{f,\alpha}$.

Thus, resonance theorem states that a subset $E \subseteq X$ is bounded in X iff for every $f \in X^*$ there are $\alpha, \beta \in \mathbb{R}$ such that E lies between the hyperplanes $H_{f,\alpha}$ and $H_{f,\beta}$.

Using uniform boundedness principle, we can show that $\mathcal{P}[a, b]$ with $\|.\|_\infty$ is not a Banach space.

Proposition 20.

Let $X = \mathcal{P}[a, b]$ with $\|.\|_\infty$. Using uniform boundedness principle, show that X is not a Banach space.

Proof.

We construct a sequence of bounded linear operators on X which is pointwise bounded but not uniformly bounded, so that X cannot be complete. For $x(t) = \sum_{j=0}^{\infty} \alpha_j t^j$ ($\alpha_j = 0$ for $j > N_x$), $\|x\| = \max_j |\alpha_j|$. Define $f_n : X \to \mathbb{K}$ by $f_n(x) = \alpha_0 + \alpha_1 + \cdots + \alpha_{N_x}$. Then each f_n is linear and bounded since $|\alpha_j| \leq \|x\|$, so that $|f_n(x)| \leq (n+1)\|x\|$. Hence (f_n) is pointwise bounded.

We now show that (f_n) is not uniformly bounded, that is, there is no c such that $\|f_n\| \leq c$ for all n. This we do by choosing particularly disadvantageous polynomials. For f_n we choose x defined by $x(t) = 1 + t + \cdots + t^n$. Then $\|x\| = 1$ and $\|f_n\| \geq \frac{|f_n(x)|}{\|x\|} = n + 1$ so that $(\|f_n\|)$ is unbounded.

Summary

Theorem 21.

Let $\mathcal{A}$ be a subset of bounded linear operators from a Banach space X into a normed space Y. The following statements are equivalent:

1. *$\mathcal{A}$ is bounded (uniformly bounded).*
2. *$\{Tx : T \in \mathcal{A}\}$ is bounded for each $x \in X$ (pointwise bounded).*
3. *$\{y^*(Tx) : T \in \mathcal{A}\}$ is bounded for each $x \in X$ and $y^* \in Y^*$ (weakly bounded).*

Theorem 22 (Continuity of bilinear maps).

Let X, Y, Z be normed spaces, of which X or Y is Banach and $A : X \times Y \to Z$ is linear. Define $A_x : Y \to Z$ and $A^y : X \to Z$ by $A_x(y) = A(x, y) = A^y(x)$, $x \in X, y \in Y$. If $A_x \in \mathcal{B}(Y, Z)$ for all $x \in X$ and $A^y \in \mathcal{B}(X, Z)$ for all $y \in Y$, then A is jointly continuous, and $\|A(x, y)\| \leq k\|x\| \, \|y\|$ for $x \in X, y \in Y$, where k is a constant. In particular, if $x_n \to x$ in X and $y_n \to y$ in Y, then $A(x_n, y_n) \to A(x, y)$ in Z.

Proof. Suppose that X is Banach. Let $E = \{y \in Y : \|y\| \leq 1\}$. Consider the family $\{A^y : y \in E\}$ of bounded linear operators from X to Z. For $x \in X$, $y \in E$, $\|A^y(x)\| = \|A(x, y)\| = \|A_x(y)\| \leq \|A_x\| \, \|y\| \leq \|A_x\|$. This shows that this family is pointwise bounded on X. Hence the uniform boundedness principle implies that $\{\|A^y\| : y \in E\}$ is a bounded set. Let $\|A^y\| \leq k$ for all $y \in E$. For $x \in X, y \in Y, y \neq 0$ let $z = \frac{y}{\|y\|}$. Then $z \in E$, and so, $\|A(x, z)\| = \|A^z(x)\| \leq \|A^z\| \, \|x\| \leq k\|x\|$. However, $A(x, z) = A_x(z) = \frac{1}{\|y\|} A(x, y)$. Hence, $\|A(x, y)\| = \|y\| \, \|A(x, z)\| \leq k\|x\|\|y\|$. Since $A(x, 0) = A_x(0) = 0$, this is true when $y = 0$ also. Therefore, $\|A(x, y)\| \leq k\|x\|\|y\|$ for all $x \in X$, $y \in Y$. We can easily prove that A is jointly continuous. If Y is Banach, we can consider the family $\{A_x : x \in X, \|x\| \leq 1\}$ of bounded linear operators from Y to Z and proceed as above.

Proposition 23.

Let (a_n) be a sequence of scalars such that $\sum_{n=0}^{\infty} a_n x_n$ converges for every (x_n) converging to zero. Then $\sum_{n=0}^{\infty} |a_n| < \infty$ (the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent).

Proof.

Let $f_n : c_0 \to \mathbb{K}$ by $f_n(x) = \sum_{i=1}^{n} a_i x_i$, for $x = (x_n) \in c_0$. Since $|f_n(x)| \leq \sum_{i=1}^{n} |a_i|.\|x\|_\infty$, each f_n is a continuous linear functional on c_0. Let $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sum_{i=1}^{n} a_i x_i$. Then f is a continuous linear functional on X and $\|f\| + \sum_{n=1}^{\infty} |a_n|$. Since f is bounded, $\sum_{n=1}^{\infty} |a_n| < \infty$.

Proposition 24.

Let $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A sequence $y = (y_1, y_2, \ldots)$ belongs to ℓ_q iff $\sum_{j=1}^{\infty} x_j y_j$ converges for every $x \in \ell_p$.

Particular cases for $p = 1$ and $p = \infty$ of the Proposition 24 is discussed below.

Exercise 25.

Let (a_n) be a sequence of scalars such that $\sum_{n=1}^{\infty} a_n x_n$ converges for every $(x_n) \in \ell_1$. Show that $(a_n) \in \ell_\infty$.

Exercise 26.

Let $f_n : \ell_1 \to \mathbb{K}$ by $f_n(x) = \sum_{i=1}^{n} a_i x_i$. If $\sum a_k x_k$ is convergent, whenever $(a_n) \in c_0$, then $(a_n) \in \ell_1$.

Convergence of Sequences of Operators

Sequences of bounded linear operators and functionals arise frequently in the abstract formulation of concrete situations, for instance in connection with convergence problems of Fourier series or sequences of interpolation polynomials or methods of numerical integration, to name just a few.

In such cases one is usually concerned with the convergence of those sequences of operators or functionals, with boundedness of corresponding sequences of norms or with similar properties. For sequences of operators $T_n \in \mathcal{B}(X, Y)$ there types of convergence turn out be of theoretical as well as practical value.

These are

1. Convergence in the norm on $\mathcal{B}(X, Y)$,
2. Strong convergence of $(T_n x)$ in Y,
3. Weak convergence of $(T_n x)$ in Y.

The definitions and terminology were introduced by J. von Neumann.

Theorem 27 (Uniform Boundedness Theorem).

Let X and Y be normed spaces and (T_n) be a sequence in $\mathcal{B}(X, Y)$ such that $(T_n x)$ converges in Y for every $x \in X$. If $(\|T_n\|)$ is a bounded sequence, then the operator $T : X \to Y$ defined by $Tx = \lim_n T_n x$, $x \in X$, belongs to $\mathcal{B}(X, Y)$ and $\|T\| \leq \liminf_n \|T_n\|$.

Proof.

Define $Tx = \lim T_n x$. The linearity of T follows from its definition. Also, for every $x \in X$, we have
$\|Tx\| = \lim_{n\to\infty} \|T_n x\| = \liminf_n \|T_n x\| \leq \liminf_n \|T_n\| \, \|x\|$ so that T is bounded and $\|T\| \leq \liminf_n \|T_n\|$.

Uniform Boundedness Theorem

In the above result, it was required that $(T_n x)$ converges for every $x \in X$. One may ask whether this condition can be guaranteed by knowing the convergence of $(T_n x)$ for x in some subset E of X.

It can be easily seen that if $E \subseteq X$ is such that $span(E) = X$ or if E contains any of the sets $\{x \in X : \|x\| < r\}$, $\{x \in X : \|x\| = r\}$ for some $r > 0$, then the convergence of $(T_n x)$ for $x \in E$ implies the convergence of $(T_n x)$ for all $x \in X$.

Now we show that if Y is a Banach space, then E can be any set such that $span(E)$ is dense in X.

The strict inequality can occur in $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$.

Example 28.

If $X = \ell_1$ *and* $f_n(x) = x_n$*, then* $f_n \in X^*$ *and* $f_n(x) \to f(x) = 0$ *for each* $x \in X$*. Since* $\|f_n\| = 1$ *for each* n*, we have* $0 = \|f\| < \liminf \|f_n\| = 1$*.*

Remark 29.

By Uniform Boundedness Theorem, the condition of boundedness of $(\|T_n\|)$ *is redundant if* X *is Banach. That is, if* X *is Banach and if* $T_n \to_s T$*, then* T *is also bounded by Uniform Boundedness principle. What is the relation between this* T *and the uniform limit of* (T_n)*? From above point, the norm of* T *is less than or equal to the norm of uniform limit of* (T_n)*.*

Strong Cauchy Sequence

Suppose X is Banach. If a sequence (T_n) is Cauchy in the strong sense, that is, for all $x \in X$ the sequence $(T_n x)$ is Cauchy in X, then there exists $T \in \mathcal{B}(X)$ such that $T_n \to T$ strongly. That is, if X is complete, then the strong operator topology on $\mathcal{B}(X, Y)$ is complete.

The sequence (T_n) in $\mathcal{B}(X, Y)$ is said to be a strong Cauchy sequence if the sequence $(T_n x)$ is a Cauchy sequence for all $x \in X$. The above result says that **if the spaces X and Y are Banach spaces, then $\mathcal{B}(X, Y)$ is complete in the strong sense.**

Theorem 30.

Let $T_n \in \mathcal{B}(X, Y)$ and Y be Banach. If $T_n x \to Tx$ for every x in a total subset E and $(\|T_n\|)$ is bounded, then $T_n \to_s T$. Moreover, if X and Y are Banach, $T_n \to_s T$ iff $T_n \to_s$ on some total subset.

Proof.

Suppose Y is Banach and $T_n x \to Tx$ for every $x \in E$ such that $D = \langle E \rangle$ is dense in X. Let $x \in X$.

Since D is dense in X, for each $\varepsilon > 0$, there exists $u \in D$ such that $\|x - u\| < \varepsilon$. Since $(T_n u)$ converges for all $u \in D$, there exists N such that $\|T_n u - T_m u\| < \varepsilon$, for all $n, m \geq N$.

Since $(\|T_n\|)$ is bounded, there exists $c > 0$ such that $\sup_n \|T_n\| \leq c$. For $n, m \in N$,

$\|T_n x - T_m x\| \leq (\|T_n\| + \|T_m\|)\|x - u\| + \|T_n u - T_m u\| \leq (2c + 1)\varepsilon$, $(T_n x)$ is Cauchy in Y,

hence $(T_n x)$ converges, for all $x \in X$.

Strong Convergence

To use the Banach-Stienhauss theorem, the following result on the strong convergence of a sequence of bounded linear operators is useful.

Theorem 31.

Suppose X and Y are Banach. A sequence of operators $T_n \in \mathcal{B}(X, Y)$ converges strongly to an operator $T \in \mathcal{B}(X, Y)$ if and only if the sequence $(T_n x)$ converges for any x in a total subset E.

Proof. If $(T_n x)$ converges for any $x \in X$, then $(T_n x)$ converges for any x from a dense subset of X. From Banach-Steinhaus theorem, there exists $c > 0$ such that $\|T_n\| \leq c$. On the other hand, assume that $(T_n x)$ converges for any x in a dense subspace M of X and $\|T_n\| \leq c$. For any $x \in M$ define the operator T_0 via $T_0 x = \lim_n T_n x$. It is clear that the operator T_0 is linear and moreover from $\|T_n x\| \leq c\|x\|$ it follows that $\|T_0\| \leq c$. We extend the operator T_0 on the whole space X as follows: for any $x \in X$ take $x_m \in M$, $x_m \to x$ and set $Tx = \lim_m T_0 x_m$. It is easy to check that the limit exists, does not depend on the choice of the sequence x_m and $\|T\| \leq c$. Let us show that for any $x \in X$, $T_n x \to Tx$. For $\varepsilon > 0$ and take $x_0 \in M$ such that $\|x - x_0\| < \frac{\varepsilon}{4(c+1)}$ and next take $N_0 > 0$ such that for any $n > N_0$, $\|T_n x_0 - Tx_0\| < \frac{\varepsilon}{2}$. Thus, for any $n > N_0$, we obtain $\|T_n x - Tx\| \leq (\|T_n\| + \|T\|)(\|x_0 - x\|) + \|T_n x - Tx_0\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Strong Convergence

In the above proof, it is sufficient to consider $(T_n x)$ is Cauchy in Y in place of $(T_n x)$ converges for every x in a total subset M of X.

Theorem 32.

A sequence (T_n) of operators $T_n \in \mathcal{B}(X, Y)$, where X and Y are Banach spaces, is strongly operator convergent iff the sequence $(\|T_n\|)$ is bounded and the sequence $(T_n x)$ is Cauchy in Y for every x in a total subset M of X.

Proof. We prove only the converse part. Let $x \in X$ and $\varepsilon > 0$ be given. Since *span* M is dense in X, there is a $y \in$ *span* M such that $\|x - y\| < \frac{\varepsilon}{3c}$, where $\|T_n\| \leq c$ for all n. Since $y \in$ *span* M, the sequence $(T_n y)$ is Cauchy. Hence there is an N such that $\|T_n y - T_m y\| < \frac{\varepsilon}{3}$, for all $m, n > N$. Using these two inequalities and applying the triangle inequality, we readily see that $(T_n x)$ is Cauchy in Y because for $m, n > N$ we obtain $\|T_n x - T_m x\| \leq \|T_n x - T_n y\| + \|T_n y - T_m y\| + \|T_m y - T_m x\| < \|T_n\| \, \|x - y\| + \frac{\varepsilon}{3} + \|T_m\| \, \|x - y\| < \varepsilon$. Since Y is complete, $(T_n x)$ converges in Y. Since $x \in X$ was arbitrary, this proves strong operator convergence of (T_n).

Applications in summability of sequences and numerical integration

Corollary 33.

A sequence (f_n) of bounded linear functionals on a Banach space X is weak* convergent, and the limit being a bounded linear functional on X iff the sequence $(\|f_n\|)$ is bounded and the sequence $(f_n(x))$ is Cauchy for every x in a total subset M of X.

This has interesting applications in summability of sequences and numerical integration.

An Interesting Fact

X and Y are normed spaces. (T_n) is a sequence of bounded linear operators from X into Y. Suppose $T_n \to_s T$ where $Tx = \lim T_n x$. Since addition and scalar multiplication are continuous, T is linear. Suppose X is Banach. Then T is bounded but T_n need not converge to T in the norm topology.

Example 34.

Let $T_n : \ell_2 \to \ell_2$ by $T_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. $T_n x \to Ix$ because $\|T_n x - x\|_2^2 = \sum_{j=n+1}^{\infty} |x_j|^2$ as $n \to \infty$ but $\sup_{\|x\| \leq 1} \|T_n x - x\|_2 \geq \|T_n e_{n+1} - e_{n+1}\|_2 = 1$ for all $n \in \mathbb{N}$. We may observe that for each $n \in \mathbb{N}$, the operators T_n and $I - T_n$ are orthogonal projections, and hence, we already know that $\sup_{\|x\| \leq 1} \|T_n x - x\|_2 = \|T_n - I\| = 1$ for all n. This example shows that $T_n \in \mathcal{B}(X, Y)$ and $T_n x \to x$ for all x. Since X is Banach, I is bounded but $T_n \not\to I$ in the norm topology.

Useful consequence of the uniform boundedness principle in numerical functional analysis

A consequence of the uniform boundedness principle, which is very useful in numerical functional analysis, is the following.

Theorem 35.

Let X be a Banach space, Y a normed space, and (A_n) be a sequence in $\mathcal{B}(X, Y)$ such that $(A_n x)$ converges for every $x \in X$. Let $A : X \to Y$ be defined by $Ax = \lim_n A_n x$, $x \in X$. Then for every totally bounded subset $S \subseteq X$, $\sup_{x \in S} \|A_n x - Ax\| \to 0$ as $n \to \infty$.

We obtained uniform convergence of a sequence of operators in $\mathcal{B}(X, Y)$ on totally bounded subsets of X provided we have pointwise convergence on a Banach space X.

Gliding Hump Argument

We now see a different derivation of the uniform boundedness principle that does not use any form of the Baire category theorem. The argument, essentially due to Hahn, is of a type called a "gliding hump" (also called "sliding hump") argument.

The only use of completeness in the argument is to assure that a certain absolutely convergent series converges.

The proof of uniform boundedness principle without using any other form of the Baire category theorem is essentially from Hahn's 1922 paper, through he stated the result only for sequences of linear functionals. This is called a *gliding hump argument*.

Gliding Hump Argument

Gliding hump arguments probably first appeared in work by Henri Lebesgue from 1905. Hahn specifically stated in his paper that the basic method for his proof was taken from a 1909 paper by Lebesgue.

The original proofs given by Hans Hahn and Stefan Banach in 1922 were quite different: they began from the assumption that

$$\sup_{T \in \mathcal{F}} \|T\| = \infty$$

and used a "gliding hump" (also called "sliding hump") technique to construct a sequence (T_n) in $\mathcal{F}$ and a point $x \in X$ such that

$$\lim_{n \to \infty} \|T_n x\| = \infty.$$

A non-Baire proof of Banach-Steinhaus theorem

We now see a proof of the uniform boundedness theorem which can be comprehended by observing a single equation. The proof is elementary in Halmos's sense of the work in that it does not use the Baire category theorem or any related lemmas. It uses a so-called "gliding-hump" technique. It is weaker than the Baire-based proof since the other one shows that an unbounded family of operators can only be pointwise bounded on a meager set of points, wheras this proof reveals only that some sequence may be constructed on which an unbounded family of operators is unbounded at some point.

Theorem 36 (Banach-Steinhaus Theorem).

Let X be a Banach space and Y be a normed space and $\mathcal{F} \subseteq B(X, Y)$. Then if $\sup\{\|Tx\| : T \in \mathcal{F}\} < \infty$ for all $x \in X$ we must have that

$$\sup\{\|T\| : T \in \mathcal{F}\} < \infty.$$

Proof adapted from "A Short Course in Banach Space Theory", by N.L.Carothers.

Suppose that $\mathcal{F}$ is not uniformly bounded, i.e. $\sup_{T \in F} \|T\| = \infty$.

We wish to establish the existence of a point at which $\mathcal{F}$ is not bounded.

Fix $0 < \delta < \frac{1}{2}$. Select T_1 from F Let x_1 in X be so $\|x_1\| = \delta$ and $\|T_1 x_1\| > (1-\delta)\|T_1\| \, \|x_1\|$.

We now conduct an induction. Having selected $T_1, \ldots, T_{n-1}$ and $x_1, \ldots, x_{n-1}$, select T_n from $\mathcal{F}$ for which

$$\|T_n\| > \frac{M_{n-1} + n}{(1-2\delta)\delta^n},$$

where $M_{n-1} = \sup_{T \in \mathcal{F}} \|T(x_1 + \cdots + x_{n-1})\|$ and then choose x_n in X_0 with $\|x_n\| = \delta^n$ and $\|T_n x_n\| > (1-\delta)\|T_n\| \, \|x_n\| = (1-\delta)\delta^n \|T_n\|$.

Proof (contd...)

Notice that the series $\sum_{k=1}^{\infty} x_k$ has Cauchy sequence of partial sums, hence converges in the Banach space X. Observe that the choices of T_n and x_n entail that

$$\left(1 - \frac{\delta}{1-\delta}\right) \|T_n x_n\| = \frac{1-2\delta}{1-\delta} \|T_n x_n\| > (1-2\delta)\delta^n \|T_n\| > M_{n-1} + n$$

while

$$\left\| T_n \sum_{k=n+1}^{\infty} x_k \right\| \leq \|T_n\| \sum_{k=n+1}^{\infty} \delta^k = \|T_n\| \frac{\delta^{n+1}}{1-\delta} < \frac{\delta}{1-\delta} \|T_n x_n\|.$$

We put this together to compute for $x = \sum_{k=1}^{\infty} x_k$ that

$$\begin{aligned} \|T_n x\| &\geq \|T_n x_n\| - \left\| T_n \sum_{k=1}^{n-1} x_k \right\| - \left\| T_n \sum_{k=n+1}^{\infty} x_k \right\| \\ &> \left(1 - \frac{\delta}{1-\delta}\right) \|T_n x_n\| - M_{n-1} > n. \end{aligned}$$

Hence F is not pointwise bounded on all of X which contradicts the assumption. Thus the proof is completed.

Humps for T_n

Notice that the point of this proof is that we may write

$$T_n x = \underbrace{T_n \sum_{k=1}^{n-1} x_k}_{norm \leq M_{n-1}} + \underbrace{T_n x_n}_{norm >> M_{n-1}} + \underbrace{T_n \sum_{k=n+1}^{\infty} x_k}_{norm << \|T_n x_n\|}$$

so that the growth of $T_n x_n$ drives the growth of $T_n x$. The series defining x "humps", for T_n, at n, and is relatively tame otherwise; it uniformly sums bad phenomena for all T_n, simultaneously. In building the proof, we selected vectors x_n to be summable via a geometric series (probably primarly because these are the only sequences we really understand), and choose the growth of operators T_n, afterwards.

Refer page 49 in the book “An Introduction to Banach Space Theory” by Robert E. Megginson

Suppose that is a nonempty collection of bounded linear operators from a Banach space X into a normed space Y such that $\sup\{\|T\| : T \in\} = +\infty$. The goal is to find an x in X such that $\sup\{\|Tx\| : T \in\} = +\infty$.

(a) The proof is based on the existence of sequences (T_n) and (x_n) in and X respectively such that the following two conditions are satisfied for each positive integer n:

$$\|T_n x_n\| \geq n + \sum_{j=1}^{n-1} \|T_n x_j\| \text{ (or } \geq 1 \text{ if } n = 1);$$

$$\|x_n\| \leq 2^{-n} \min\{\|T_j\|^{-1} : j < n\} \text{ (or } \leq 2^{-1} \text{ if } n = 1).$$

Argue that the only obstacle to the inductive construction of the sequence is the existence of an x with the desired property. The existence of such a pair of sequences may therefore be assumed.

Exercise 37.

(b) *Show that the series* $\sum_n x_n$ *converges to some* x *in* X*.*

(c) *Show that* $\sum_{j=n+1}^{\infty} \|T_n x_j\| \leq 1$ *for each* n*.*

(d) *Show that* $\|T_n x\| \geq n - 1$ *for each* n*, so* $\sup\{\|Tx\| : T \in\} = +\infty$*.*

Proof

The sequences (T_n) and (x_n) are chosen so that $\sum_n x_n$ converges to some x in X and, for each n, the major contribution to $T_n x$ comes from $T_n x_n$.

If $(\|T_n x_j\|)_{j=1}^{\infty}$ is considered to be a sequence that depends on the parameter n, then the sequence has a "hump" in it at the n^{th} term.

As n increases, this hump glides forward and has unbounded height. Incidentally, the argument can be simplified very slightly by replacing

$$\sum_{j=1}^{n-1} \|T_n x_j\|$$

by $\|T_n(\sum_{j=1}^{n-1} x_j)\|$ in (a), but then it is not obvious that $\|T_n x_n\|$ must be the dominant term of the sequence $(\|T_n x_j\|)_{j=1}^{\infty}$, and therefore it is harder to see the hump.

A simple elementary proof of the uniform boundedness theorem[1]

Theorem 38 (Uniform Boundedness Theorem).

Let $\mathcal{F}$ be a family of bounded linear operators from a Banach space X to a normed linear space Y. If $\mathcal{F}$ is pointwise bounded (i.e., $\sup_{T \in \mathcal{F}} \|Tx\| < \infty$ for all $x \in X$), then $\mathcal{F}$ is norm-bounded (i.e., $\sup_{T \in \mathcal{F}} \|T\| < \infty$).

[1] Alan D. Sokal, A really simple elementary proof of the uniform boundedness theorem, The American Mathematical Monthly, Vol.118, No.5 (May 2011), pp.450-452

Proof

To prove the theorem, we first prove the following lemma.

Lemma 39.

Let T be a bounded linear operator from a normed linear space X to a normed linear space Y. Then for any $x \in X$ and $r > 0$, we have

$$\sup_{x' \in B(x,r)} \|Tx'\| \geq \|T\| r, \tag{1}$$

where $B(x,r) = \{x' \in X : \|x' - x\| < r\}$.

Proof. For $\xi \in X$ we have

$$\max\{\|T(x+\xi)\|, \|T(x-\xi)\|\} \geq \frac{1}{2}[\|T(x+\xi)\| + \|T(x-\xi)\|] \geq \|T\xi\|, \tag{2}$$

where the second $\geq$ uses the triangle inequality in the form $\|\alpha - \beta\| \leq \|\alpha\| + \|\beta\|$. Now take the supremum over $\xi \in B(0,r)$.

Proof of the Uniform Boundedness Theorem

Suppose that $\sup_{T \in \mathcal{F}} \|T\| = \infty$,and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \geq 4^n$.

Then set $x_0 = 0$, and for $n \geq 1$ use the Lemma 39 to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \leq 3^{-n}$ and $\|T_n x_n\| \geq \frac{2}{3} 3^{-n} \|T_n\|$.

The sequence (x_n) is Cauchy, hence covergent to some $x \in X$; and it is easy to see that $\|x - x_n\| \leq \frac{1}{2} 3^{-n}$ and hence

$$\|T_n x\| \geq \frac{1}{6} 3^{-n} \|T_n\| \geq \frac{1}{6} (4/3)^n \to \infty.$$

As just seen, this proof is most conveniently expressed in terms of a sequence (x_n) that converges to x. This contrasts with the earlier "gliding hump" proofs, which used a series that sums to x. Of course, sequences and series are equivalent, so each proof can be expressed in either language; it is a question of taste which formulation one finds simpler.

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